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Delay-dependent input-output stability conditions for non-autonomous neutral type differential equations in a Banach space

Michael Gil'

Abstract. In a Banach space we consider a class of linear non-autonomous neutral type differential equations with several delays. For the considered equations we derive explicit delay-dependent input-output stability conditions. Applications to neutral type integro-differential equations are also discussed.

1. Introduction and statement of the main result

This paper is devoted to a class of linear non-autonomous neutral type differential equations with several variable delays, whose coefficients are bounded operators in a Banach space. Such equations include, in particular, various neutral type integro-differential equations.

The basic method for the stability analysis of the neutral type functional differential equations in a Hilbert space is the generalized Lyapunov-Krasovskii method, cf. [13, 16, 18, 20] and references given therein. By that method many great results have been obtained, however, to the best of our knowledge, the stability of neutral type nonautonomous equations in a Banach space with several delays are not investigated in the available literature. Below we obtain explicit delay-dependent input-output stability conditions for the considered equations. Note that the literature on the delay-dependent stability criteria is rather rich, but mainly equations in a finite dimensional space are considered, cf. [1, 5–7, 10, 11, 14, 15].

Introduce the notations: $\mathcal{X}$ is a complex Banach space with a norm $\|\cdot\|_{\mathcal{X}} = \|\cdot\|$ and the unit operator $I_{\mathcal{X}} = I$. By $\mathcal{B}(\mathcal{X})$, we denote the set of all bounded linear operators in $\mathcal{X}$. For any $A \in \mathcal{B}(\mathcal{X})$, $\sigma(A)$ is the spectrum and $\|A\|$ is the operator norm. Below the continuity and differentiability are understood in the strong sense. $C([a, b], \mathcal{X})$ is the space of

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$\mathcal{X}$-valued functions f defined and continuous on a finite or infinite segment $[a,b]$ and equipped with the finite norm

$$\|f\|_{C(a,b)} = \|f\|_{C([a,b],\mathcal{X})} = \sup_{a\le t\le b} \|f(t)\|_{\mathcal{X}}.$$

For simplicity, we will denote $R^1 = (-\infty,\infty), R_+ = [0,\infty)$. Let $B_j(t) : R_+ \to \mathcal{B}(\mathcal{X})$ $(j=1,\dots,m_E)$ be continuous, $B(t,\tau) : R_+ \times [0,1] \to \mathcal{B}(\mathcal{X})$ be continuous in t and piece-wise continuous in τ, and $T \in \mathcal{B}(\mathcal{X})$.

The present paper is devoted to the equation

$$\begin{gathered} w'(t) - Tw'(t-\eta) = (Ew)(t) + f(t), \\ (0<\eta<\infty;\ f\in C(R_+,\mathcal{X}),\ t\ge 0), \end{gathered} \tag{1}$$

where

$$(Ew)(t) = \sum_{k=1}^{m_E} B_k(t)w(t-h_k(t)) + \int_0^1 B(t,s)w(t-h_0(s))ds$$

and $h_k(t)$, $(k=1,\dots,m_E<\infty)$, are continuous nonnegative functions defined on R_+, such that $h_k(t)\le\eta$, $(t\ge0)$; $h_0(s)$ is a continuous nonnegative function defined on $[0,1]$, such that $h_0(s)\le\eta$, $(0\le s\le 1)$.

Take the zero initial condition

$$w(t)\equiv 0, \quad (t\le 0). \tag{2}$$

A solution of problem (1), (2) is an $\mathcal{X}$-valued continuous function w defined on $(-\infty,\infty)$, having a continuous derivative for all $t>0$ and satisfying (1) and (2).

Below we check the existence of solutions under consideration. Equation (1) is said to be *input-output stable*, if there is a positive constant m_0 independent of $f\in C(R_+,\infty)$, such that

$$\sup_{t\ge0}\|w(t)\|_{\mathcal{X}} \le m_0\|f\|_{C(R_+,\mathcal{X})}$$

for a solution $w(t)$ of problem (1), (2). As is well-known, the input-output is deeply connected with some other types of stabilities [9].

Throughout the paper it is assumed that

$$\|T\| < 1 \tag{3}$$

and

$$\chi(E) := \sup_{t\ge0}\left(\sum_{k=1}^{m_E}\|B_k(t)\| + \int_0^1\|B(t,\tau)\|d\tau\right) < \infty, \tag{4}$$

therefore

$$\psi(E) := \sup_{t\ge0}\left(\sum_{k=1}^{m_E}\|B_k(t)\|h_k(t) + \int_0^1\|B(t,\tau)\|h_0(\tau)d\tau\right) < \infty.$$

Finally, we introduce the operators $M(t)$ and V_M by

$$M(t) := \sum_{k=1}^{m_E} B_k(t) + \int_0^1 B(t,\tau)d\tau$$

and

$$(V_M u)(t) = \int_0^t U(t,s)u(s)ds, \quad (u \in C(R_+, \mathcal{X})),$$

where $U(t,s)$ $(t \geq s \geq 0)$ is the evolution operator of the differential equation

$$z'(t) = M(t)z(t). \tag{5}$$

Now we are in a position to formulate our main result.

Theorem 1. *Let V_M be bounded in $C(R_+, \mathcal{X})$ with a norm $\|V_M\| = \|V_M\|_{C(R_+,\mathcal{X})}$. Let the conditions* (3), (4) *and*

$$\zeta_0 := \frac{\|V_M\|_{C(R^1,\mathcal{X})}\chi(E)}{1 - \|T\|}(\|T\| + \psi(E)) < 1 \tag{6}$$

hold. Then equation (1) *is input-output stable.*

The proof of this theorem is presented in the next section. Theorem 1 is sharp in the following sense: if $T = 0$ and $\psi(E) = 0$, then equation (1) takes the form (5). In this case condition (6) is automatically holds, if V_M is bounded. But the boundedness of V_M is necessary for the stability.

Note that our stability conditions are based, in particular, on the norm estimates for V_M. In Section 3 we recall such estimates, assuming that $M(t)$ is dissipative or satisfies the so called generalized Lipschitz condition. In Section 4 we discuss the application of Theorem 1 to integro-differential equations.

2. Proof of Theorem 1

Extend $B_k(t)$ $(k = 1, \ldots, m_E)$ and $B(t,\tau)$ by zero to $t \in (-\infty, 0)$ and denote the extensions by the same symbols. Besides, due to (4), for the norm of E in $C(R^1)$ we have $\|E\| \leq \chi(E)$. Rewrite (1) as

$$\frac{d}{dt}\big(w(t) - (Sw)(t)\big) = (Ew)(t) + f(t), \quad (t \geq 0). \tag{7}$$

Here $(Sw)(t) = Tw(t-\eta)$. Integrating (7) from $-\infty$ to t with (2) taken into account and extending f to $(-\infty, 0)$ by zero, we have

$$w(t) - (Sw)(t) = \int_0^t (Ew)(s)ds + f_1(t), \quad (t \geq 0), \tag{8}$$

where

$$f_1(t) = \int_0^t f(s)ds, \quad (t \geq 0)$$

and

$$f_1(t) = 0, \quad (t \leq 0).$$

Operators E and S are bounded on space $C(R^1)$ and map it into itself. Besides, due to (3)

$$(I-S)^{-1}=\sum_{k=0}^{\infty} S^k \quad \text{and} \quad \|(I-S)^{-1}\|_{C(R^1)} \le (1-\|T\|)^{-1}.$$

For a finite $t_0 > \eta$, introduce the subspace

$$C_0 := \{g \in C(-\infty, t_0) : g(t) = 0, \quad (t \in (-\infty, 0])\}.$$

Define on C_0 the operator W by

$$(Wg)(t) = \int_{-\infty}^{t} (E(I-S)^{-1}g)(s)ds, \quad (t \ge 0)$$

and

$$(Wg)(t) = 0, \quad (t \le 0, \ g \in C_0).$$

Simple calculations show that

$$\|W^k g\|_{C(-\infty,t_0)} = \|W^k g\|_{C(0,t_0)} \le \frac{1}{k!}(t_0)^k \|E(I-S)^{-1}\|^k_{C(R^1)},$$

for $g \in C_0$, $\|g\|_{C(0,t_0)} = 1$. Clearly, $f_1 \in C_0$. Thus,

$$(I-W)^{-1} = \sum_{k=0}^{\infty} W^k$$

and the equation

$$u - Wu = f_1$$

has a unique solution $u \in C_0$, and

$$u'(t) = (E(I-S)^{-1}u)(t) + f(t), \quad (-\infty < t \le t_0).$$

So $w = (I-S)^{-1}u$ satisfies equation (8) and $w' = (I-S)^{-1}u'$. Since (8) is equivalent to (1), we have proved the *existence of solutions to problem* (1), (2).

Furthermore, for a finite $t_0 > \eta$ for the brevity put $|w|_{t_0} = \|w\|_{C(-\infty,t_0)}$. Then (1) and (3) imply

$$|w'|_{t_0} \le \|T\||w'|_{t_0} + \chi(E)|w|_{t_0} + \|f\|_{C(R_+)}.$$

Hence,

$$|w'|_{t_0} \le \gamma |w|_{t_0} + c_1 \|f\|_{C(R_+)}, \quad (c_1 = \text{constant} > 0), \tag{9}$$

where $\gamma := \chi(E)(1-\|T\|)^{-1}$.

We can write $(Ew)(t) = M(t)w(t) + (Zw)(t)$, $(t \ge 0)$, where

$$(Zw)(t) = \sum_{k=1}^{m} B_k(t)\big(w(t-h_k(t)) - w(t)\big) + \int_0^1 B(t,\tau)\big(w(t-h_0(\tau)) - w(t)\big)d\tau.$$

Thus (1) can be written as

$$w'(t) = (Sw')(t) + M(t)w(t) + (Zw)(t) + f(t), \quad (t \geq 0).$$

Hence, by the Variation of Constants formula, a solution of problem (1), (2) satisfies the equation

$$w(t) = \int_0^t U(t,s)\big((Zw)(s) + f(s) + (Sw')(s)\big)ds, \quad (t > 0). \tag{10}$$

Observe that

$$w(t) - w(t-\tau) = \int_{t-\tau}^t w'(s)ds, \quad (t > \tau > 0).$$

Hence, by (9)

$$\|w(t) - w(t-\tau)\|_{C(0,t_0)} \leq \tau |w'|_{t_0} \leq \tau\gamma |w|_{t_0} + c_2\|f\|_{C(R_+)},$$

with $c_2 = c_1\eta$. Thus

$$\begin{aligned} |Zw|_{t_0} \leq \sup_{0\leq t\leq t_0} &\Bigg(\sum_{k=1}^m \|B_k(t)\|\ \|w(t-h_k(t)) - w(t)\| \\ &+ \int_0^1 \|B(t,\tau)\|\ \|w(t-h_0(\tau)) - w(t)\|d\tau\Bigg) \\ \leq \sup_{t\geq 0}&\Bigg(\sum_{k=1}^m \|B_k(t)\|h_k(t)(\gamma|w|_{t_0} + c_2\|f\|_{C(R_+)}) \\ &+ \int_0^1 \|B(t,\tau)\|h_0(\tau)(\gamma|w|_{t_0} + c_2\|f\|_{C(R_+)})d\tau\Bigg), \end{aligned}$$

i.e.,

$$|Zw|_{t_0} \leq \gamma\psi(E)|w|_{t_0} + c_3\|f\|_{C(R_+)}), \tag{11}$$

where c_3 does not depend on f. From (10) and (11) it follows

$$\begin{aligned} |w|_{t_0} &\leq \|V_M\|_{C(R_+)}|Sw' + Zw + f|_{t_0} \\ &\leq \|V_M\|_{C(R_+)}\big(|Sw'|_{t_0} + |Zw|_{t_0} + \|f\|_{C(R_+)}\big) \\ &\leq \|V_M\|_{C(0,R_+)}\big[\|T\||w'|_{t_0} + \gamma\psi(E)|w|_{t_0} + (c_3+1)\|f\|_{C(R_+)}\big]. \end{aligned}$$

Now (9) yields

$$|w|_{t_0} \leq \|V_M\|_{C(R_+)}\gamma|w|_{t_0}(\|T\| + \psi(E)) + c_0\|f\|_{C(R_+)},$$

where c_0 is a constant independent of f. But

$$\gamma\|V_M\|_{C(R_+)}(\|T\|+\psi(E)) = \|V_M\|_{C(R_+)}\chi(E)(1-\|T\|)^{-1}(\|T\|+\psi(E)) = \zeta_0.$$

Therefore,

$$|w|_{t_0} \leq \zeta_0|w|_{t_0} + c_0\|f\|_{C(R_+)}.$$

If (6) holds, then

$$|w|_{t_0} \leq \frac{c_0}{1-\zeta_0}\|f\|_{C(R_+)}.$$

Hence, letting $t_0 \to \infty$, we get

$$\|w\|_{C(R_+)} \le \frac{c_0}{1-\zeta_0}\|f\|_{C(R_+)}.$$

This proves the required result. □

3. ESTIMATES FOR THE NORM OF V_M

We need the following lemma.

Lemma 1. *Let there be a real Riemann-integrable function $\nu(t)$, such that*

$$\|I + M(t)\delta\| \le 1 + \nu(t)\delta + o(\delta), \quad (t \ge 0), \tag{12}$$

for all sufficiently small $\delta > 0$. Then

$$\|U(t,s)\| \le \exp[\int_s^t \nu(s_1)ds_1], \quad (t \ge s \ge 0).$$

For the proof see, for example, [12, Lemma 3.1]. Assume that

$$\theta_M := \sup_{t\ge 0}\int_0^t \exp\left[\int_s^t \nu(s_1)ds_1\right] ds < \infty.$$

Then due to Lemma 1

$$\|V_M\|_{C(R_+)} \le \theta_M,$$

provided condition (12) holds.

Let $\mathcal{X} = \mathcal{H}$ be a Hilbert space and $\Lambda(M_R(t)) = \sup\ \sigma(M_R(t))$, where $M_R(t) = \frac{1}{2}(M(t) + M^*(t))$ and the asterisk means the adjointness. Since

$$\begin{aligned}\|(I + M(t)\delta)h\|^2 &= \|(I + M(t)\delta)h\|^2 \\ &= \|(I + 2M_R(t)\delta + M^*(t)M(t)\delta^2)h\| \\ &\le 1 + 2\Lambda(M_R(t))\delta + o(\delta), \quad (h \in \mathcal{H},\ \|h\| = 1),\end{aligned}$$

we can take $\nu(t) = \Lambda(M_R(t))$. Hence we arrive at the Wintner inequality

$$\|U(t,s)\| \le \exp[\int_s^t \Lambda(M_R(s_1))ds_1], \quad (t \ge s \ge 0),$$

cf.[8, Theorem III.4.7].

Now assume that $M(t)$ satisfies the generalized Lipschitz condition

$$\|M(t) - M(\tau)\| \le a(|t-\tau|), \quad (t,\tau \ge 0), \tag{13}$$

where $a(t)$ is a positive piece-wise continuous function defined on $[0,\infty)$. A particular case of (13) is the traditional Lipschitz condition

$$\|M(t) - M(\tau)\| \le q_0|t-\tau|, \quad (q_0 = \text{constant} > 0;\ t,\tau \ge 0).$$

In addition to (3.4) suppose that there is a positive integrable on $[0,\infty)$ function $p(t)$ independent of s, such that

$$\|\exp[M(s)t]\| \le p(t), \quad (t,s\ge 0) \quad \text{and} \quad J_0 := \int_0^\infty p(t)dt < \infty. \tag{14}$$

Lemma 2. *Let the conditions* (13), (14) *and*

$$\xi := \int_0^\infty a(s)p(s)ds < 1$$

hold. Then

$$\|V_M\|_{C(R_+)} \le \frac{J_0}{1-\xi}.$$

For the proof see Corollary 4.2 from [12]. About other bounds for the norm of V_M see, for instance, [12].

4. Example

Consider the following partial neural type integro-differential equation

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} - a(x)\frac{\partial u(t-\eta,x)}{\partial t} &= c(t,x)u(t-h_1(t),x) \\ &+ \int_0^x k(t,x,x_1)u(t-h_2(t),x_1)dx_1 + f(t,x), \end{aligned} \tag{15}$$

where $t>0$, $0\le x\le 1$, $a(.)$ and $c(.,.)$ are real continuous functions defined on $[0,1]$ and $[0,\infty)\times[0,1]$, respectively, $c(.,.):[0,\infty)\times[0,1]\to\mathbb{R}$ $k(.,.,.):[0,\infty)\times[0,1]^2\to\mathbb{R}$ has the following property: the integral $\int_0^1|k(t,x,x_1)|dx_1$ is continuous in t and x, and $f(t,x)$ is also continuous in t and x.

Consider equation (15) in space $C(0,1)$ of scalar continuous functions defined on $[0,1]$ with the sup-norm. Equation (15) has the form (1) with $m_E=2$; $h_1(t),h_2(t)$ are the same as above; $B(t,\tau)=0$, $(Tz)(x)=a(x)z(x)$,

$$\big(B_1(t)z\big)(x) = c(t,x)z(x), \quad \big(B_2(t)z\big)(x) = \int_0^1 k(t,x,x_1)z(x_1)dx_1,$$

for $z\in C(0,1)$ and $M(t)=B_1(t)+B_2(t)$. Condition (3) takes the form

$$\hat a := \sup_{x\in[0,1]} |a(x)| < 1. \tag{16}$$

Condition (4) takes the form

$$\chi_1 = \sup_{t\ge 0}\,(\sup_x |c(t,x)| + \|B_2(t)\|) < \infty. \tag{17}$$

So, in the considered case $\|T\|=\hat a$, $\chi(E)=\chi_1$ and $\psi(E)=\psi_1$, where

$$\psi_1 = \sup_{t\ge 0}\,(h_1(t)\sup_x |c(t,x)| + h_2(t)\|B_2(t)\|).$$

Assume that

$$\begin{aligned}\|I + M(t)\delta\| &= \max_x |z(x) + c(t,x)z(x)\delta + (B_2(t)z)(x)\delta| \\ &\le 1 + \nu_1(t)\delta + o(\delta), \quad (z \in C(0,1);\ \|z\| = 1),\end{aligned} \tag{18}$$

where $\nu_1(t)$ is a continuous function, i.e., condition (3.1) is fulfilled. If, in addition,

$$\Lambda_1 := \sup_{t\ge 0} \nu_1(t) < 0, \tag{19}$$

then according to (12), $\|V_M\| \le \frac{1}{|\Lambda_1|}$.

Now Theorem 1 implies the following corollary.

Corollary 1. *Let the conditions* (16)-(19) *and*

$$\frac{\chi_1(\hat{a} + \psi_1)}{1 - \hat{a}} < |\Lambda_1|$$

hold. Then equation (15) *is input-output stable.*

About other recent approaches to the neutral type integro-differential equations see, for instance, the papers [2–4, 17, 19, 21].

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Michael Gil'
Department of Mathematics
Ben Gurion University of the Negev
P.O. Box 653,
Beer-Sheva 84105
Israel
E-mail address: gilmi@bezeqint.net